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BIMODULES, SPECTRA, AND FELL BUNDLES*

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ABSTRACT

Let E be a Fell bundle over an r-discrete principal groupoid Γ that is measurewise amenable with respect to every quasi-invariant measure on $\Gamma^{(0)}$. Let $C^*(E)$ be the corresponding C^*-algebra and let $C_0(E^0)$ be the diagonal subalgebra. The Spectral Theorem on Bimodules proved in this paper gives a description of $C_0(E^0)$-bimodules in $C^*(E)$.

1. Introduction

Roughly speaking, a Fell bundle over a groupoid is a generalization of the situation that arises when one partitions the $N \times N$ matrices, $M_N(\mathbb{C})$, into block matrices indexed by, say, $n \times n$. The fiber over the point (i, j), $1 \leq i, j \leq n$, is the collection of $k_i \times k_j$ matrices, and $\sum_{i=1}^{n} k_i = N$. Our interest in this note, specialized to this setting, is to understand the structure of the linear subspaces of $M_N(\mathbb{C})$ that are bimodules over the subalgebra of block diagonal matrices, i.e., over $\sum_{i=1}^{n} \oplus M_{k_i}(\mathbb{C})$. In particular, we are interested in the structure of the **subalgebras** of $M_N(\mathbb{C})$ that contain $\sum_{i=1}^{n} \oplus M_{k_i}(\mathbb{C})$. Our results generalize a number of other results found in the literature (see in particular [12, 13, 11] and [7]).

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Our basic setup is this: Throughout this note, Γ will denote an r-discrete, second countable, locally compact, Hausdorff, principal groupoid in the sense of Renault [14] and we always assume that the range map $r\colon \Gamma \to \Gamma^{(0)}$ is a local homeomorphism. As pointed out in [14], this is equivalent to assuming that Γ has a Haar system (which necessarily is given by counting measure on the r-fibers). We denote the Haar system by $\{\lambda^u\}_{u\in\Gamma^{(0)}}$. Our main result requires that Γ is **amenable** in the sense that it is measurewise amenable with respect to every quasi-invariant measure on $\Gamma^{(0)}$, as defined in [14]. However, we shall not invoke this assumption until it is used. Also, E will denote a Fell bundle over Γ in the sense of Kumjian [9].* This means, first of all, that E is a Banach bundle over Γ. We shall follow the standard references, [3], [4], [5], and [6], for the material we need about Banach bundles. We note, in particular, that a Banach bundle for us means what Dupré and Gillette call an (F) Banach bundle and this is essentially the same thing as Dixmier's continuous field of Banach spaces, thanks to the theorem of Douady and Dal Soglio-Hérault [5, Appendix]. We shall assume that our bundles are separable in the sense that the set $C_c(E)$ of continuous, compactly supported sections of E endowed with the inductive limit topology is second countable [6]. We write $E(\gamma)$ for $p^{-1}(\gamma)$, where $p\colon E \to \Gamma$ is the bundle projection. Then to be a Fell bundle it is required that E satisfy the following additional points.

1. There is a partially defined multiplication from $E^{(2)} := \{(a,b)|(p(a),p(b)) \in \Gamma^{(2)}\}$ to E which is bilinear; associative, whenever the relevant products are defined; and has the property that the bundle map p is a homomorphism.

2. The product is continuous from $E^{(2)}$ to E, where $E^{(2)}$ is given the relative topology, and satisfies the inequality $\|ab\| \le \|a\|\|b\|$, for $(a,b) \in E^{(2)}$, where the norms denote the norms on the various fibers involved.

3. The bundle E is **saturated** in the sense that $E(\alpha)E(\beta)$ is total in $E(\alpha\beta)$ for all pairs $(\alpha,\beta) \in \Gamma^{(2)}$.

4. There is an involution on E, denoted $a \to a^*$, such that $E(\gamma)^* = E(\gamma^{-1})$, $(a^*)^* = a, (ab)^* = b^*a^*, a^*a \ge 0$ in $E(s(p(a)))$, and $\|a^*a\| = \|a\|^2$, for all $(a,b) \in E^{(2)}$.

* Fell bundles over groupoids are called C^*- algebras over groupoids in [18]. This paper has been in circulation as a preprint for a number of years, but has not appeared in print. We will therefore follow [9] as our basic reference.

Implicit in these points is the fact that for each unit $u \in \Gamma^{(0)}$, $E(u)$ is a C^*-algebra and each $E(\gamma)$ is endowed with $E(r(\gamma))$- and $E(s(\gamma))$-valued inner products, making $E(\gamma)$ a left Hilbert C^*-module over $E(r(\gamma))$ and a right Hilbert C^*-module over $E(s(\gamma))$. The saturation condition guarantees that, in fact, $E(\gamma)$ is an $E(r(\gamma)), E(s(\gamma))$-**imprimitivity** bimodule. In particular, $E(\gamma)^* = E(\gamma^{-1})$ is naturally isomorphic to the conjugate $E(s(\gamma)), E(r(\gamma))$-bimodule. We shall therefore make this identification throughout.

The space $C_c(E)$ becomes a $*$-algebra under the operations

$$f * g(\gamma) = \sum_{\gamma=\alpha\beta} f(\alpha)g(\beta)$$
$$= \int f(\alpha)g(\alpha^{-1}\gamma)d\lambda^{r(\gamma)}(\alpha)$$

and

$$f^*(\gamma) = f(\gamma^{-1})^*,$$

$f, g \in C_c(E)$ and $\alpha, \beta, \gamma \in \Gamma$. A certain completion of $C_c(E)$, that we shall define in a moment, is a C^*-algebra that we shall denote $C^*(E)$. Each element of $C^*(E)$ is a continuous section of E that vanishes at infinity on Γ. Consequently, it is possible to think of $C^*(E)$ as an algebra of block matrices, where the blocks are indexed by Γ. The block sizes vary from point to point of Γ. Indeed, if Γ is the trivial groupoid $\{1, 2, \cdots, n\}^2$ and if E is the bundle over Γ whose fiber over (i, j) is space of $k_i \times k_j$ complex matrices, we find that $C^*(E)$ is the algebra $M_N(\mathbb{C})$, $N = \sum_{i=1}^n k_i$, with which we began our discussion.

Let E^0 denote the restriction of E to $\Gamma^{(0)}$. Then, as we have noted, E^0 is a C^*-bundle over $\Gamma^{(0)}$. We write $C_0(E^0)$ for the C^*-algebra of C_0-cross sections of E^0 on $\Gamma^{(0)}$. Since $\Gamma^{(0)}$ is a clopen subset of Γ, by virtue of the fact that Γ is r-discrete, $C_0(E^0)$ is a sub-C^*-algebra of $C^*(E)$. Of course, $C_0(E^0)$ should be viewed as a C^*-algebra of block diagonal matrices. Our primary objective in this note is to identify the subspaces $B \subseteq C^*(E)$ that are invariant under left and right multiplication by elements in $C_0(E^0)$, i.e. that are $C_0(E^0)$-bimodules in $C^*(E)$. This, in turn, is used to identify the closed subalgebras of $C^*(E)$ that contain $C_0(E^0)$. To state our theorem, we introduce the following notation. We write $\mathcal{F}$ for the set of all closed subbundles F of E (in the sense of [4, p. 21 ff.]) with the property that for every $\gamma \in \Gamma$, $F(\gamma)$ is a (closed) $E(r(\gamma))$-$E(s(\gamma))$ subbimodule of $E(\gamma)$. We make no assumption about F being closed under multiplication or under the involution. We write $\mathcal{A}$ for the subbundles $A \in \mathcal{F}$ that contain E^0 and **are** closed under multiplication: if $(a, b) \in A^{(2)} := (A \times A) \cap E^{(2)}$, then $ab \in A$. Further, if $F \in \mathcal{F}$, we write $B(F)$ for the closure of the set of all sections

$f \in C_c(F)$. Evidently, every $B(F)$ is a (closed) bimodule over $C_0(E^0)$ in $C^*(E)$. The Spectral Theorem for Bimodules is the converse assertion.

THEOREM 1 (Spectral Theorem for Bimodules): *If Γ is amenable, then every closed bimodule over $C_0(E^0)$ in $C^*(E)$ is of the form $B(F)$ for a unique bundle $F \in \mathcal{F}$. Further, $B(F)$ is an algebra containing $C_0(E^0)$ if and only if $F \in \mathcal{A}$.*

The proof is broken into a series of steps. Most of our efforts focus on the first assertion. We attend to the second at the end of the paper. Basically, our goal is to use Kumjian's Stabilization Theorem (Corollary 4.5 of [9]), which shows that $C^*(E)$ is strongly Morita equivalent to the C^*-algebra of a groupoid crossed product, to reduce Theorem 1 to Theorem 15.18 of [7]. The problems to be overcome involve a careful interplay between bundles, various cross sectional spaces, tensor products of Hilbert C^*-modules, and Haagerup tensor products.

2. Kumjian's Stabilization Theorem

To state this theorem, as well as to give the full definition of $C^*(E)$, observe that if for $u \in \Gamma^{(0)}$, we set

$$V(u) = \sum_{s(\gamma)=u} {}^{\oplus} E(\gamma)$$

with the usual, direct sum, $E(u)$-valued inner product, then $V(u)$ becomes a Hilbert C^*-module over $E(u)$. The family, V, of the $V(u)$ forms a Hilbert C^*-module bundle over $\Gamma^{(0)}$. To identify a family of cross sections of this bundle that defines the Banach bundle structure on V, we use $C_c(E)$. Each $f \in C_c(E)$ defines a section ξ_f via the formula: $\xi_f(u) := \sum_{s(\gamma)=u}^{\oplus} f(\gamma)$. To see that $\sum_{s(\gamma)=u}^{\oplus} f(\gamma)$ lies in $V(u)$, simply note that $f^*f(u) = \sum_{u=\alpha\beta} f^*(\alpha)f(\beta) = \sum_{u=s(\gamma)} f(\gamma)^* f(\gamma)$. Thus, using [5, Proposition 10.4] and [4, Proposition 1.3], it is easy to check that $C_c(E)$ determines a Banach bundle structure on V making it a Hilbert C^*-module bundle. In fact, as Kumjian proves in Section 3 of [9] the space $C_0(V)$ of continuous cross sections of V vanishing at infinity becomes a Hilbert $C_0(E^0)$-module that is the completion of $C_c(E)$ in the $C_0(E^0)$-valued inner product: $\langle \xi_f, \xi_g \rangle(u) = f^* * g(u)$. There is a natural, faithful, $*$-representation π of $C_c(E)$ in the C^*-algebra of continuous, adjointable operators on $C_0(V)$, $\mathcal{L}(C_0(V))$, given by the formula $\pi(f)\xi_g = \xi_{f*g}$. The closure of $\pi(C_c(E))$ in $\mathcal{L}(C_0(V))$ is the desired C^*-algebra $C^*(E)$. Strictly speaking, this should be called the **reduced** C^*-algebra of E, and denoted $C^*_{\mathrm{red}}(E)$, but in the situation in which we are working, we do not have to make that distinction. We will take this point up again later in the context of crossed products.

The algebras $K(V(u)), u \in \Gamma^{(0)}$, form a bundle of C^*-algebras over $\Gamma^{(0)}$ that we denote by $K(V)$. The cross sections of $K(V)$ used to define the topology on $K(V)$ are the rank-1 operator sections, $\xi_f \otimes \xi_g^*$, $f, g \in C_c(E)$. It is not hard to see that $K(C_0(V))$ is naturally isomorphic to $C_0(K(V))$.

We pause to insert the following lemma that summarizes some of the salient features of our discussion to this point and that will be used later. The details of the proof that have not already been developed are easy to supply, and so will be omitted.

LEMMA 2: *The space $C_0(V)$ is a $C_0(K(V)) - C_0(E^0)$-imprimitivity bimodule.*

The groupoid Γ acts on $K(V)$. To define the action requires a little preparation. Kumjian notes in paragraph 3.3 of [9] that $V(r(\gamma)) \otimes_{E(r(\gamma))} E(\gamma)$ is naturally isomorphic to $V(s(\gamma))$ via the map

$$\Big(\sum_{s(\alpha)=r(\gamma)} e_\alpha\Big) \otimes e \to \sum_{s(\alpha)=r(\gamma)} e_\alpha e,$$

where $e_\alpha \in E(\alpha)$ and $e \in E(\gamma)$. The product $e_\alpha e$ is in $E(\alpha\gamma)$ and the map is easily seen to be isometric. The saturation assumption on E guarantees that this is map is surjective. Also, since $E(\gamma)^* = E(\gamma^{-1})$ coincides with the conjugate module of $E(\gamma)$, we may write the conjugate of $V(u)$ as $V(u)^* = \sum_{u=r(\gamma)} E(\gamma)$. We then find that $E(\gamma) \otimes_{E(s(\gamma))} V(s(\gamma))^*$ is isomorphic to $V(r(\gamma))^*$. To define the action of Γ on $K(V)$, fix $\gamma \in \Gamma$. Choose α with $s(\alpha) = r(\gamma)$, choose β with $s(\beta) = s(\gamma)$, and pick $a \in E(\alpha)$, $b \in E(\beta)$, and $c \in E(\gamma)$. Then elements of the form $ac \otimes b^*$ span a dense subset of $K(V(s(\gamma)))$, while elements of the form $a \otimes cb^*$ span a dense subset of $K(V(r(\gamma)))$. One checks that the map σ_γ from $K(V(s(\gamma)))$ to $K(V(r(\gamma)))$ defined by the equation

$$\sigma_\gamma(ac \otimes b^*) = a \otimes cb^*$$

extends to a C^*-isomorphism. To check that σ defines a continuous action of Γ on $K(V)$, one forms the pull back bundles $s_*(K(V))$ and $r_*(K(V))$ on Γ and views σ as a map σ_* from $s_*(K(V))$ to $r_*(K(V))$ in the obvious way. The continuity of σ amounts to the fact that σ_* maps continuous sections for $s_*(K(V))$ to continuous sections of $r_*(K(V))$. This is easy to verify in the present circumstance.

With $K(V)$ and the action σ, we form a bundle over Γ, denoted $K(V) \times_\sigma \Gamma$, but which is really the pull-back bundle $s_*(K(V))$. That is $K(V) \times_\sigma \Gamma := \{(k, \gamma) | \gamma \in \Gamma, k \in K(V(s(\gamma)))\}$. The projection map is $p(k, \gamma) = \gamma$. The (partially defined) product on $K(V) \times_\sigma \Gamma$ is given by the formula

$$(k_1, \gamma_1) \cdot (k_2, \gamma_2) = (\sigma_{\gamma_2^{-1}}(k_1)k_2, \gamma_1\gamma_2)$$

and the involution is given by the formula

$$(k,\gamma)^* = (\sigma_\gamma(k^*), \gamma^{-1}).$$

The verification that $K(V) \times_\sigma \Gamma$ is a Fell bundle over Γ is straightforward.

THEOREM 3 (Kumjian's Stabilization Theorem [9, Corollary 4.5]): *The C^*-algebras $C^*(K(V) \times_\sigma \Gamma)$ and $C^*(E)$ are strongly Morita equivalent.*

What is important for us is the form of the equivalence bimodule found by Kumjian, that we denote by $\mathcal{X}$. The way he arrives at $\mathcal{X}$ is to form the groupoid $\Gamma \times \Delta$, where Δ is the transitive groupoid, $\{0,1\}^2$ on a two point space $\{0,1\}$. He extends the bundle E on Γ to a bundle D on $\Gamma \times \Delta$. The fibers of D are given by the following formulae:

$$\begin{aligned}
D(\gamma,(0,0)) &= E(\gamma),\\
D(\gamma,(1,0)) &= V(r(\gamma)) \otimes_{E(r(\gamma))} E(\gamma),\\
D(\gamma,(0,1)) &= E(\gamma) \otimes_{E(s(\gamma))} V(s(\gamma))^*,\\
D(\gamma,(1,1)) &= K(V) \times_\sigma \Gamma.
\end{aligned}$$

He shows that D is a Fell bundle over $\Gamma \times \Delta$, which is a groupoid of the same kind as Γ, and takes $\mathcal{X} = p^\perp C^*(D)p$ where p is the projection in the multiplier algebra of $C^*(D)$ which is the characteristic function of $\Gamma^{(0)} \times (0,0)$.

We want to think about $\mathcal{X}$ directly, without reference to the groupoid $\Gamma \times \Delta$. It is a space of C_0-cross sections to the Banach bundle $r_*(V) \otimes E$. More explicitly, the fiber over γ in $r_*(V)$ is $V(r(\gamma))$ while the fiber over γ in E is $E(\gamma)$. So, the fiber over γ in $r_*(V) \otimes E$ is $V(r(\gamma)) \otimes_{E(r(\gamma))} E(\gamma)$. This is the Hilbert C^*-module tensor product taken over the C^*-algebra $E(r(\gamma))$. As we have noted before, $V(r(\gamma)) \otimes_{E(r(\gamma))} E(\gamma)$ is naturally isomorphic to $V(s(\gamma))$ as an $E(s(\gamma))$-module, but we want to maintain a distinction at the moment. The cross sections determining the topology on $r_*(V) \otimes E$ may of course be taken to be pointwise elementary tensor products of cross sections of $r_*(V)$ and E. We write these as $\xi \cdot f$ where $\xi \in C_0(V)$, $f \in C_c(E)$ and $\xi \cdot f(\gamma) = \xi(r(\gamma)) \otimes f(\gamma)$("$=$"$\xi(r(\gamma))f(\gamma)$). Note that we are taking $\xi \in C_0(V)$ even though $\xi \circ r$ is not in $C_0(r_*(V))$. The point is that $\xi \cdot f$ is continuous with compact support on Γ because f has this property. Observe that the linear span $\mathcal{X}_0$ of such sections carries a $C^*(E)$-valued inner product defined on elements of the form $\xi \cdot f$ by the formula

$$\langle \xi \cdot f, \eta \cdot g\rangle(\gamma) = f^* * \langle \xi, \eta\rangle * g(\gamma), \tag{1}$$

where $\xi, \eta \in C_0(V)$, and $f, g \in C_c(E)$, and where $\langle \xi, \eta\rangle$ denotes the value of the $C_0(E^0)$-valued inner product on $C_0(V)$. The defining formula for the $C^*(E)$-

valued inner product on $\mathcal{X}_0$ makes sense, since $C_0(E^0)$ is a subalgebra of $C^*(E)$. The space $\mathcal{X}$ is the completion of $\mathcal{X}_0$ in the norm associated with this inner product. Thus, $\mathcal{X}$ is the internal tensor product $C_0(V) \otimes_{C_0(E^0)} C^*(E)$ realized in terms of the bundle $r_*(V) \otimes E$. It is now easy to see that $\mathcal{X}$ described in this way is the same as the $\mathcal{X}$ that Kumjian produces using $\Gamma \times \Delta$.

We are using the general fact [10, Proposition 4.5] that if $\mathcal{E}$ is a Hilbert C^*-module over a C^*-algebra A, if $\mathcal{F}$ is a Hilbert C^*-module over a C^*-algebra B, and if φ: $A \to \mathcal{L}(\mathcal{F})$ is a C^*-representation, making $\mathcal{F}$ a left A-module, then $\mathcal{E} \otimes_A \mathcal{F}$ is in a natural way a Hilbert B-module. The B-valued inner product on $\mathcal{E} \otimes_A \mathcal{F}$ is given by a formula analogous to equation (1). Now it is important for our considerations that the operator space structure on $\mathcal{E} \otimes_A \mathcal{F}$ is that of the **module Haagerup tensor product** over A, $\mathcal{E} \otimes_{hA} \mathcal{F}$, as was proved by Blecher in [1, Theorem 4.3]. That is, $\mathcal{E} \otimes_A \mathcal{F}$ and $\mathcal{E} \otimes_{hA} \mathcal{F}$ are completely isometrically isomorphic operator space modules over the C^*-algebra B. Thus, we may summarize our discussion in this section as

LEMMA 4: *With the notation established above, we have the following isomorphisms:*

(1) $C_0(V) \otimes_{hC_0(E^0)} C^*(E) \simeq C_0(V) \otimes_{C_0(E^0)} C^*(E) \simeq \mathcal{X}$, *and*

(2) $C^*(E) \otimes_{hC_0(E^0)} C_0(V^*) \simeq C^*(E) \otimes_{C_0(E^0)} C_0(V^*) \simeq \mathcal{X}^*$,

where the first tensor product in each formula is the Haagerup module tensor product over $C_0(E^0)$, the second is the interior module tensor product, and the modules $C_0(V^*)$ and $\mathcal{X}^*$ are the conjugate modules of $C_0(V)$ and $\mathcal{X}$, respectively.

Proof: The proof of the first assertion was developed above. The second follows from the first, and the second half of Theorem 4.3 in [1]. ∎

This lemma, Corollary 4.5 of [9], and Corollary 4.7 of [1] combine to show that $C_0(V) \otimes_{hC_0(E^0)} C^*(E) \otimes_{hC_0(E^0)} C_0(V^*)$ has the structure of a C^*-algebra that is naturally isomorphic to $C^*(K(V) \times_\sigma \Gamma)$. However, we want to realize this isomorphism at the level of bundles. For this, consider the bundle $r_*(V) \otimes E \otimes s_*(V^*)$ over Γ. It has fiber $V(r(\gamma)) \otimes_{E(r(\gamma))} E(\gamma) \otimes_{E(s(\gamma))} V(s(\gamma))^*$, $\gamma \in \Gamma$, with an obvious family of cross sections. It carries the structure of a Fell bundle where the multiplication is defined by the formula

$$(\xi_1 \otimes e_1 \otimes \eta_1^*)(\xi_2 \otimes e_2 \otimes \eta_2^*) = \xi_1 \otimes e_1 \langle \eta_1, \xi_2 \rangle e_2 \otimes \eta_2^*,$$

for $(\xi_i \otimes e_i \otimes \eta_i) \in V(r(\gamma_i)) \otimes_{E(r(\gamma_i))} E(\gamma_i) \otimes_{E(s(\gamma_i))} V(s(\gamma_i))^*$, with $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$, and the involution is defined by the formula

$$(\xi \otimes e \otimes \eta^*)^* = \eta \otimes e^* \otimes \xi^*.$$

Thus $r_*(V) \otimes E \otimes s_*(V^*)$ is the bundle that Kumjian calls F in paragraph 4.3 of [9]. Define $\mu: r_*(V) \otimes E \otimes s_*(V^*) \to K(V) \times_\sigma \Gamma$ by the formula

$$\mu(\xi \otimes e \otimes \eta^*) = (\xi e \otimes \eta^*, \gamma),$$

where $\xi \otimes e \otimes \eta^* \in V(r(\gamma)) \otimes_{E(r(\gamma))} E(\gamma) \otimes_{E(s(\gamma))} V(s(\gamma))^*$. Recall that $\xi e \in V(s(\gamma))$ so that μ is well defined. In fact, as Kumjian shows in Proposition 4.4 of [9], μ is a Fell bundle isomorphism from $r_*(V) \otimes E \otimes s_*(V^*)$ onto $K(V) \times_\sigma \Gamma$. Now write μ^* for the map that μ induces on sections. Thus, for $\xi, \eta \in C_0(V)$, and $f \in C_c(E)$, we have $\mu^*(\xi \otimes f \otimes \eta^*)(\gamma) = \mu(\xi(r(\gamma)) \otimes f(\gamma) \otimes \eta^*(s(\gamma))) = (\xi(r(\gamma))f(\gamma) \otimes \eta^*(s(\gamma)), \gamma)$.

PROPOSITION 5: *The map μ^* is a $C_0(K(V))$-isomorphism from $C_0(V) \otimes_{hC_0(E^0)} C^*(E) \otimes_{hC_0(E^0)} C_0(V^*)$ onto $C^*(K(V) \times_\sigma \Gamma)$.*

Proof: It is clear that μ^* is a $*$-algebra isomorphism that takes a generating set of sections of $r^*(V) \otimes E \otimes s^*(V^*)$ to a generating set of sections of $K(V) \times_\sigma \Gamma$. It is also clear that μ^* is a $C_0(K(V))$-module map. Thus to prove that μ^* is isometric, we need only write the following string of isometries and note that μ^* is their composition:

$$\begin{aligned} C^*(K(V) \times_\sigma \Gamma) &\cong \mathcal{X}^* \otimes_{C^*(E)} \mathcal{X} \\ &\cong \mathcal{X}^* \otimes_{hC^*(E)} \mathcal{X} \\ &\cong \big(C_0(V) \otimes_{hC_0(E^0)} C^*(E)\big) \otimes_{hC^*(E)} \big(C^*(E) \otimes_{hC_0(E^0)} C_0(V^*)\big) \\ &\cong C_0(V) \otimes_{hC_0(E^0)} \big(C^*(E) \otimes_{hC^*(E)} C^*(E)\big) \otimes_{hC_0(E^0)} C_0(V^*) \\ &\cong C_0(V) \otimes_{hC_0(E^0)} C^*(E) \otimes_{hC_0(E^0)} C_0(V^*). \end{aligned}$$

The passage from the first line to the second is based on Theorem 4.3 of [1]. The next line follows from Lemma 4. The line after follows from Theorem 2.6 of [2]. The last line is a consequence of Lemma 2.5 of [2]. ∎

We may reverse the roles of $C^*(E)$ and $C^*(K(V) \times_\sigma \Gamma)$ to conclude that $C_0(V^*) \otimes_{hC_0(K(V))} C^*(K(V) \times_\sigma \Gamma) \otimes_{hC_0(K(V))} C_0(V)$ is naturally isomorphic to $C^*(E)$. However, we want to do this at the level of bundles. To this end, observe first that if A and B are C^*-algebras and if $\mathcal{E} = {}_A\mathcal{E}_B$ is an imprimitivity bimodule linking A and B, then the map from $A \otimes_A \mathcal{E}$ to $\mathcal{E}$ given by sending $a \otimes e$ to ae is a (completely) isometric A, B-bimodule isomorphism. We call this map ν_1. By virtue of the fact that $\mathcal{E}$ is an A,B-imprimitivity bimodule, $\mathcal{E}^* \otimes_A \mathcal{E}$ is identified with B through the map ν_2 that sends $\xi^* \otimes \eta$ to $\langle \xi, \eta \rangle_B$. Also, we write ν_3 for the imprimitivity bimodule isomorphism that sends $\mathcal{E} \otimes_B B$ to $\mathcal{E}$ through the formula

$e \otimes b \to eb$. Finally, passing to the setting of the situation under discussion in this note, we write ν_4 for the **inverse** of the isomorphism, noted above, that sends $E(\gamma) \otimes_{E(s(\gamma))} V(s(\gamma))^*$ to $V(r(\gamma))^*$ by the formula $e \otimes \eta^* \to e\eta^*$. Composing these, we obtain a Fell bundle isomorphism ν from $r_*(V)^* \otimes (K(V) \times_\sigma \Gamma) \otimes s_*(V)$ to E. Explicitly, $\nu = \nu_3 \circ (I \otimes \nu_2) \circ (I \otimes I \otimes \nu_1) \circ (\nu_4 \otimes I \otimes I)$, where we have identified $(K(V) \times_\sigma \Gamma)(\gamma)$ with $K(s(\gamma))$. Again, we write ν_* for the isomorphism that ν induces on sections.

PROPOSITION 6: *The map ν_* is a $C_0(E^0)$-isomorphism from $C_0(V^*) \otimes_{hC_0(K(V))} C^*(K(V) \times_\sigma \Gamma) \otimes_{hC_0(K(V))} C_0(V)$ onto $C^*(E)$.*

Proof: The proof is as before: ν_* acts on a dense $*$-algebra of sections in $C_0(V^*) \otimes_{hC_0(K(V))} C^*(K(V) \times_\sigma \Gamma) \otimes_{hC_0(K(V))} C_0(V)$, carrying them to a dense $*$-algebra of sections in $C^*(E)$. The fact that ν_* is isometric follows from the fact that it is expressed as the composition of isometries using the following string of equivalences:

$$\begin{aligned}
&C_0(V^*) \otimes_{hC_0(K(V))} C^*(K(V) \times_\sigma \Gamma) \otimes_{hC_0(K(V))} C_0(V)\\
&\cong C_0(V^*) \otimes_{hC_0(K(V))} \big(C_0(V) \otimes_{hC_0(E^0)} C^*(E) \otimes_{hC_0(E^0)} C_0(V^*)\big)\\
&\qquad\qquad \otimes_{hC_0(K(V))} C_0(V)\\
&\cong \big(C_0(V^*) \otimes_{hC_0(K(V))} C_0(V)\big) \otimes_{hC_0(E^0)} C^*(E)\\
&\qquad\qquad\qquad\qquad\qquad \otimes_{hC_0(E^0)} \big(C_0(V^*) \otimes_{hC_0(K(V))} C_0(V)\big)\\
&\cong C_0(E^0) \otimes_{hC_0(E^0)} C^*(E) \otimes_{hC_0(E^0)} C_0(E^0) \cong C^*(E). \qquad \blacksquare
\end{aligned}$$

3. Subbundles and subbimodules

We have two bundles over Γ: the bundle E and the bundle $K(V) \times_\sigma \Gamma$. We want to use $r_*(V) \otimes E$ to map subbundles of E to subbundles of $K(V) \times_\sigma \Gamma$. This will enable us to reduce the proof of Theorem 1 for general Fell bundles to Fell bundles arising as crossed products.

Recall that we are writing $\mathcal{F}$ for the set of all subbundles F of E such that for every $\gamma \in \Gamma$, $F(\gamma)$ is a closed $E(r(\gamma))$-$E(s(\gamma))$ subbimodule of $E(\gamma)$. It is important to keep in mind that each $F(\gamma)$ inherits an $E(r(\gamma))$-valued inner product and an $E(s(\gamma))$-valued inner product. However, in general, neither of these is full.

Similarly, we let $\mathcal{L}$ be the set of all subbundles L of $K(V) \times_\sigma \Gamma$ such that for every $\gamma \in \Gamma$, $L(\gamma)$ is a closed $(K(V) \times_\sigma \Gamma)(r(\gamma))$-$(K(V) \times_\sigma \Gamma)(s(\gamma))$ subbimodule of $(K(V) \times_\sigma \Gamma)(\gamma)$. Since we may identify $(K(V) \times_\sigma \Gamma)(\gamma)$ with $K(V(s(\gamma)))$, such a bimodule is really a two-sided ideal in $K(V(s(\gamma)))$. We define maps Φ and Φ' between the sets $\mathcal{F}$ and $\mathcal{L}$ as follows: For $F \in \mathcal{F}$, $\Phi(F)$ is defined to be

$$r_*(V) \otimes F \otimes s_*(V^*);$$

that is, at the level of fibers, $\Phi(F)(\gamma)$ is given by the formula

$$\Phi(F)(\gamma) = V(r(\gamma)) \otimes_{E(r(\gamma))} F(\gamma) \otimes_{E(s(\gamma))} V(s(\gamma))^*.$$

On the other hand, for $L \in \mathcal{L}$, $\Phi'(L) = r_*(V^*) \otimes L \otimes s_*(V)$, so that $\Phi'(F)(\gamma)$ is given by the formula

$$\Phi'(F)(\gamma) = V(r(\gamma))^* \otimes_{K(V(r(\gamma)))} L(\gamma) \otimes_{K(V(s(\gamma)))} V(s(\gamma)).$$

In these two definitions, the tensor products of modules are interior Hilbert C^*-bimodule tensor products.

The maps Φ and Φ' are evidently well defined and Φ carries $\mathcal{F}$ to $\mathcal{L}$, while Φ' carries $\mathcal{L}$ to $\mathcal{F}$. In order to make this assertion, of course, we must identify $V(u) \otimes_{E(u)} V(u)^*$ with $K(V(u))$ and $V(u)^* \otimes_{K(V(u))} V(u)$ with $E(u)$, $u \in \Gamma^{(0)}$, and then we must recall that $K(V(r(\gamma))) \otimes_{K(V(r(\gamma)))} L(\gamma) \otimes_{K(V(s(\gamma)))} K(V(s(\gamma)))$ is canonically, completely isometrically, isomorphic to $L(\gamma)$. Similarly, we must identify $V(u)^* \otimes_{K(V(u))} V(u)$ with $E(u)$, $u \in \Gamma^{(0)}$ and note that $E(r(\gamma)) \otimes_{E(r(\gamma))} E(\gamma) \otimes_{E(s(\gamma))} E(s(\gamma))$ is canonically isomorphic to $E(\gamma)$. With these identifications and the analysis to this point, the following proposition is immediate.

PROPOSITION 7: *The maps Φ and Φ' are inverse to one another; i.e.,*

$$\Phi \circ \Phi' = \mathrm{id}_{\mathcal{L}}; \qquad \Phi' \circ \Phi = \mathrm{id}_{\mathcal{F}}.$$

Let $\mathcal{B}$ be the set of all closed $C_0(E^0)$-bimodules in $C^*(E)$ and let $\mathcal{C}$ be the set of all closed $C_0(K(V))$-bimodules in $C^*(K(V) \times_\sigma \Gamma)$. We define the following correspondences, Ψ and Ψ', between the sets $\mathcal{B}$ and $\mathcal{C}$: For every $B \in \mathcal{B}$, $\Psi(B)$ is given by the formula

$$\Psi(B) = C_0(V) \otimes_{hC_0(E^0)} B \otimes_{hC_0(E^0)} C_0(V^*),$$

while for $C \in \mathcal{C}$, $\Psi'(C)$ is given by the formula

$$\Psi'(C) = C_0(V^*) \otimes_{hC_0(K(V))} C \otimes_{hC_0(K(V))} C_0(V),$$

where all the tensor products are Haagerup module tensor products.

PROPOSITION 8: *The map Ψ is a bijection from $\mathcal{B}$ to $\mathcal{C}$, with inverse Ψ' mapping from $\mathcal{C}$ to $\mathcal{B}$, i.e.,*

$$\Psi \circ \Psi' = \mathrm{id}_{\mathcal{C}}; \qquad \Psi' \circ \Psi = \mathrm{id}_{\mathcal{B}}.$$

Proof: For $B \in \mathcal{B}$, $\Psi(B) = C_0(V) \otimes_{hC_0(E^0)} B \otimes_{hC_0(E^0)} C_0(V^*)$ is a closed subspace in $C^*(K(V) \times_\sigma \Gamma)$ which evidently is a $C_0(K(V))$-bimodule since $C_0(V)$ is a left $C_0(K(V))$-module, while $C_0(V^*)$ is a right $C_0(K(V))$-module. Thus Ψ is well defined map from $\mathcal{B}$ into $\mathcal{C}$. Similarly, one shows that Ψ' is a well defined map from $\mathcal{C}$ to $\mathcal{B}$. But

$$\Psi' \circ \Psi(B) =$$
$$(C_0(V^*) \otimes_{hC_0(K(V))} C_0(V)) \otimes_{hC_0(E^0)} B \otimes_{hC_0(E^0)} (C_0(V^*) \otimes_{hC_0(E^0)} C_0(V^*)).$$

Since $C_0(V^*) \otimes_{hC_0(K(V))} C_0(V) = C_0(V^*) \otimes_{C_0(K(V))} C_0(V)$ by Blecher's theorem [1, Theorem 4.3], and since $C_0(V^*) \otimes_{C_0(K(V))} C_0(V)$ is canonically isomorphic to $C_0(E^0)$, by Lemma 2, we conclude that

$$\Psi' \circ \Psi(B) = C_0(E^0) \otimes_{hC_0(E^0)} B \otimes_{hC_0(E^0)} C_0(E^0).$$

Since this space is canonically isomorphic to B, we conclude that $\Psi' \circ \Psi$ is the identity. A similar argument shows that $\Psi \circ \Psi' = \mathrm{id}_{\mathcal{C}}$. ∎

Given a bundle $F \in \mathcal{F}$, we set $\Xi(F) = \overline{\{f \in C_c(E) | f(\gamma) \in F(\gamma), \forall \gamma \in \Gamma\}}^{\|\cdot\|}$. The Spectral Theorem for Bimodules, Theorem 1, is the assertion that for every closed $C_0(E^0)$-bimodule B in $C^*(E)$ there exists a unique $F \in \mathcal{F}$ such that $B = \Xi(F)$, i.e., that Ξ is a bijection. The corresponding map from $\mathcal{L}$ to $\mathcal{C}$ is denoted by Ω, i.e., for $L \in \mathcal{L}$, $\Omega(L)$ is defined by the formula

$$\Omega(L) = \overline{\{f \in C_c(K(V) \times_\sigma \Gamma) | f(\gamma) \in L(\gamma), \forall \gamma \in \Gamma\}}^{\|\cdot\|}.$$

Our goal in the next section is to show that the Spectral Theorem for Bimodules is true for Ω, i.e., that for each bimodule $C \in \mathcal{C}$, there is a unique bundle $L \in \mathcal{L}$ such that $\Omega(L) = C$ (Theorem 9). Granted this result, to complete the proof of the Spectral Theorem for Bimodules in the context of general Fell bundles, we need only check that the diagram at the end of this section is commutative. That is, we want to show that $\Xi = \Psi' \circ \Omega \circ \Phi$.

To this end, we begin by noting that if $F \in \mathcal{F}$ and if $f \in C_c(E)$ lies in $\Xi(F)$, then for all $\xi \in C_0(V)$, $\eta \in C_0(V)^*$, we have $\mu^*(\xi^* \otimes f \otimes \eta) \in \Omega^{-1}(\Phi(F))$. Indeed, if $f \in \Xi(F)$, then by definition $f(\gamma) \in F(\gamma)$ for all γ. This, in turn, gives $\mu^*(\xi^* \otimes f \otimes w)(\gamma) = \xi(r(\gamma))^* \otimes f(\gamma) \otimes \eta(s(\gamma)) \in V(r(\gamma))^* \otimes F(\gamma) \otimes V(\gamma) =$

$\Phi(F)(\gamma)$ for each γ. So, consequently, $\mu^*(\xi^* \otimes f \otimes \eta) \in \Omega(\Phi(F))$. Thus, for $F \in \mathcal{F}$, $\Psi(\Xi(F)) \subseteq \Omega(\Phi(F))$. On the other hand, the same argument shows that for $L \in \mathcal{L}$, and $f \in C_c(K(V) \times_\sigma \Gamma)$, if $f \in \Omega(L)$, then for all $\xi \in C_0(V)^*$ and $\eta \in C_0(V)$, we have $\nu^*(\xi^* \otimes f \otimes \eta) \in \Xi(\Phi'(L))$. It follows from this that for $L \in \mathcal{L}$, $\Psi'(\Omega(L)) \subseteq \Xi(\Phi'(L))$. Taking $L = \Phi(F)$, for $F \in \mathcal{F}$ we get $\Psi'(\Omega(\Phi(F))) \subseteq \Xi(\Phi'(\Phi(F))) = \Xi(F)$. Applying Ψ to both sides of this inclusion we obtain $\Omega(\Phi(F)) \subseteq \Psi(\Xi(F))$. Now we just showed that $\Psi(\Xi(F)) \subseteq \Omega(\Phi(F))$, so we conclude that $\Omega(\Phi(F)) = \Psi(\Xi(F))$. Applying Ψ' to both sides of this equation, we reach the desired goal: $\Xi = \Psi' \circ \Omega \circ \Phi$.

$$\begin{tikzcd}
\begin{matrix}\text{bimodules}\\ \mathcal{B} = \{B \subset C^*(E)\}\end{matrix} \arrow[r, shift left, "\Psi"] & \begin{matrix}\text{bimodules}\\ \mathcal{C} = \{C \subset C^*(K(V) \times \Gamma)\}\end{matrix} \arrow[l, shift left, "\Psi'"] \\
\begin{matrix}\mathcal{F} = \{F \subset E\}\\ \text{subbundles}\end{matrix} \arrow[u, "\Xi"] \arrow[r, shift left, "\Phi"] & \begin{matrix}\mathcal{L} = \{L \subset K(V) \times \Gamma\}\\ \text{subbundles}\end{matrix} \arrow[u, "\Omega"'] \arrow[l, shift left, "\Phi'"]
\end{tikzcd}$$

4. The Spectral Theorem and crossed products

Our objective in this section is to prove the Spectral Theorem for Bimodules in the case of groupoid crossed products. The proof will be modeled on the proof of [11, Theorem 4.1], which involves, ultimately, a reduction to the measure theoretic context discussed by the first author in [7]. Since we will make use of Γ-sets, we will often write X for $\Gamma^{(0)}$, we shall view Γ as an equivalence relation in $X \times X$ and write elements of Γ as ordered pairs, when convenient. Thus, $r(x,y) = x$, while $s(x,y) = y$. We fix a bundle A of C^*-algebras over X and we assume that Γ acts on A continuously through a representation $\sigma\colon \Gamma \to \mathrm{Iso}(A)$. Thus, in particular, $\sigma_{(x,y)}$ is a C^*-isomorphism from $A(y)$ to $A(x)$. The crossed product Fell bundle over Γ determined by A, $A \times_\sigma \Gamma$, is $\{(a,x,y) |\ (x,y) \in \Gamma, a \in A(y)\}$. The Fell bundle structure on $A \times_\sigma \Gamma$ is defined analogously to that on $K(V) \times_\sigma \Gamma$.

Of course, in the notation established above, $C_0((A \times_\sigma \Gamma)^0) = C_0(A)$, viewed as supported on the diagonal Δ in $X \times X$. As above, we write $\mathcal{F}$ for all the closed subbundles F of $A \times_\sigma \Gamma$ (in the sense of [4, p. 21]) with the property that $F(x,y)$ is an $A(x), A(y)$-bimodule. (Remember: the multiplication is given in the product defining $A \times_\sigma \Gamma$, so $F(x,y) \subseteq A(y)$ is really a two-sided ideal over $A(y)$.) For each such bundle F, we write $B(F)$ for the closure in $C^*(A \times_\sigma \Gamma)$ of $C_c(F)$. Of course $B(F)$ is a bimodule over $C_0(A)$. Conversely, given a norm-closed subspace $B \subseteq C^*(A \times_\sigma \Gamma)$ that is a bimodule over $C_0(A)$, we define $F(B)$ to be the closed subbundle of $A \times_\sigma \Gamma$ generated by all the continuous sections of $A \times_\sigma \Gamma$ coming from B. Of course, at the outset, we do not know that there are any. However, as we shall see, if our groupoid is amenable (or equivalently, if the C^*-algebra $C^*(\Gamma)$ is nuclear [11, Theorem 3.3]), then every element of $C^*(A \times_\sigma \Gamma)$ is a C_0-section of $A \times_\sigma \Gamma$ (see Proposition 11 below). So $F(B)$ makes sense and is non-zero (if $B \neq 0$) in this case.

THEOREM 9: *In the notation just established, if Γ is amenable, then for each $C_0(A)$-bimodule $B \subseteq C^*(A \times_\sigma \Gamma)$, the inclusion*

$$C_c(F(B)) \subseteq B \subseteq C_0(F(B)) \cap C^*(A \times_\sigma \Gamma) \tag{2}$$

holds. Furthermore, the closure of $C_c(F(G))$ is $C_0(B) \cap C^*(A \times_\sigma \Gamma)$, so the right hand expression already is closed and equals B.

Of course this result applied to the bundle $K(V) \times_\sigma \Gamma$, above, is just what is needed to verify all the claims made of the map Ω.

In outline, the proof follows the analysis in [11]. However, we need to see how to modify some of the formulas in the arguments there and redo some of the lemmas. Further, we need to connect the structure of $C^*(A \times_\sigma \Gamma)$ viewed as the C^*-algebra associated with the Fell bundle $A \times_\sigma \Gamma$ with theory developed in [15]. We begin by recalling some key facts from [15], written in notation to suit our needs here.

By a representation π of $C_c(A \times_\sigma \Gamma)$ on a Hilbert space H_0, it is understood that π is a $*$-homomorphism of $C_c(A \times_\sigma \Gamma)$ into $B(H_0)$ that is continuous with respect to the inductive limit topology on $C_c(A \times_\sigma \Gamma)$ and the weak operator topology on $B(H_0)$. According to the Disintegration Theorem, [15, Theorem 4.1], such a representation determines (and is determined by) the following data: (1) a Borel Hilbert bundle H over X, with fibers denoted $H(x)$; (2) a quasi-invariant measure μ on X, in the sense of [14]; (3) a representation $U\colon \Gamma|_Y \to \mathrm{Iso}(H)$, where Y is a co-null Borel subset of X, and where $\mathrm{Iso}(H)$ denotes the groupoid of all bundle isomorphisms of H (that act isometrically between fibers), endowed with the canonical Borel structure coming from H; (4) a Borel field of representations $\{\pi_x\}_{x\in X}$, with $\pi_x\colon A(x)\to B(H(x))$; and (5) a Hilbert space isomorphism

$$W\colon H_0 \to \int_X^{\oplus} H(x)d\mu(x)$$

such that

$$U(x,y)\pi_y(\cdot)U(x,y)^{-1} = \pi_x\circ\sigma_{(x,y)}, \qquad \text{a.e.}\nu, \tag{3}$$

where ν is the measure on Γ induced by $\mu, \nu = \int \lambda^u d\mu(u)$, and such that

$$(W\pi(f)W^{-1}\xi)(x) = \int U(x,y)\pi_y(f(x,y))\Delta^{-1/2}(x,y)\xi(y)d\lambda^x(y), \tag{4}$$

for all $f\in C_c(A\times_\sigma\Gamma)$, and all $\xi\in\int_X^\oplus H(x)d\mu(x)$, where Δ is the modular function associated with μ, as in [14, Definition 3.4]. This expression for π is called the **disintegrated form** of π. Equation (3) expresses the fact that $\{\pi_x\}$ and U constitute a **covariant pair**, i.e., the analogue of a covariant representation in the context C^*-crossed products determined by group actions. We will drop reference to W in the future, and simply write π in terms of the right hand side of the above equation.

The Disintegration Theorem allows one to define a C^*-norm on $C_c(A\times_\sigma\Gamma)$ by the formula $\|f\| = \sup\{\|\pi(f)\| | \pi - \text{a representation}\}$. The point is that writing π using (4) allows one to estimate that $\|\pi(f)\|\le\|f\|_I$, where

$$\|f\|_I = \max\left\{\sup_{x\in X}\int\|f(\gamma)\|_{A(s(\gamma))}d\lambda^x(\gamma), \sup_{x\in X}\int\|f(\gamma^{-1})^*\|_{A(r(\gamma))}d\lambda^x(\gamma)\right\}. \tag{5}$$

Thus the supremum defining $\|f\|$ is finite and defines a C^*-norm on $C_c(A\times_\sigma\Gamma)$. When referring to $C^*(A\times_\sigma\Gamma)$, we shall mean the completion of $C_c(A\times_\sigma\Gamma)$ in this norm.

The relation between this norm and the one coming from viewing $C^*(A\times_\sigma\Gamma)$ as the C^*-algebra of a Fell bundle is made using those representations of $C^*(A\times_\sigma\Gamma)$ that are **induced** from representations of $C_0(A)$. Let ρ be a representation of $C_0(A)$ on a Hilbert space K. Then ρ may be disintegrated. That is, we may express K as the direct integral $\int_X^\oplus K(x)d\mu(x)$ for a measure μ, which is unique up to absolute continuity, and we may find a Borel family of representations $\{\rho_x\}_{x\in X}$ so that $\rho_x\colon A(x)\to B(K(x))$, and so that $\rho(a)\xi(x) = \rho_x(a(x))\xi(x)$, $a\in C_0(A)$ and $\xi\in\int_X^\oplus K(x)d\mu(x)$. To build the data for the representation induced by ρ, $\mathrm{Ind}(\rho)$, we begin by defining $\mathrm{Ind}(\rho_x)$ for a single point x in $X=\Gamma^{(0)}$. The Hilbert bundle for $\mathrm{Ind}(\rho_x)$ is the constant bundle over the orbit of x, $[x]$, determined by $K(x)$. Thus, the fiber over y of this bundle, $H(y)$, is $\{y\} \times K(x)$. We define $\pi_y\colon A(y) \to B(H(y))$ by $\pi_y = \rho_x \circ \sigma_{(x,y)}$, also we define $U(z,y)\colon H(y) \to H(z)$ by the formula $U(z,y)(y,\xi) = (z,\xi)$. It is straightforward to check that U and $\{\pi_x\}$ constitute a covariant pair. The quasi-invariant measure for this representation is simply the saturation of the point mass at x, ϵ_x, i.e., counting measure on $[x]$. Thus, as an easy calculation reveals, an explicit formula for $\mathrm{Ind}(\rho_x)$ is

$$(6)\quad (\mathrm{Ind}(\rho_x)(f)\xi)(z) = \sum_{u\in[x]} \rho_x \circ \sigma_{(x,u)}(f(z,u))\xi(u) = \sum_{u\in[x]} \pi_u(f(z,u))\xi(u),$$

$z \in [x]$. The representation $\mathrm{Ind}(\rho)$, then, is simply the direct integral of the representations $\mathrm{Ind}(\rho_x)$ with respect to μ. To describe this concretely, note first that r maps $s^{-1}(x)$ bijectively onto the orbit of x, $[x]$, and carries λ_x to counting measure on $[x]$. We use r to pull the bundle H defined over $[x]$ to a bundle $\tilde{H}$ defined over $s^{-1}(x)$ and to lift the data disintegrating $\mathrm{Ind}(\rho_x)$. The formulas are obvious: $\tilde{H}(y,x) = H(y) = \{y\} \times K(x)$. We define $\tilde{U}(x;(z,y))\colon \tilde{H}(y,x) \to \tilde{H}(z,x)$ by the formula $\tilde{U}((z,y);x)((y,x),\xi) = ((z,x),\xi)$, and we define $\tilde{\pi}_{(y,x)}\colon A(y) \to B(\tilde{H}(y,x)) = B(\{y\}\times K(x))$ by $\tilde{\pi}_{(y,x)}(a) = \rho_x \circ \sigma_{(x,y)}$. Then it is easy to see that $(\tilde{\pi}_{(\cdot,x)}, \tilde{U}((\cdot,\cdot);x))$ is a covariant pair unitarily equivalent, in the obvious sense, to (π, U). If we take cross sections ξ of $\tilde{H}$ of the form $\xi(y) = ((y,x), f|_{s^{-1}(x)}(y,x)\xi)$, where $f \in C_c(\Gamma)$ and $\xi \in K(x)$, we obtain a fundamental family of cross sections defining the natural Borel structure on $\tilde{H}(\cdot,x)$. The integrated form of $(\tilde{\pi}_{(\cdot,x)}, \tilde{U}((\cdot,\cdot);x))$ on $\int \tilde{H}(y,x)d\lambda_x(y)$ is then easily seen to be unitarily equivalent to $\mathrm{Ind}(\rho_x)$. It will be convenient to drop this unitary equivalence and simply rename $\mathrm{Ind}(\rho_x)$ to be the integrated form of $(\tilde{\pi}_{(\cdot,x)}, \tilde{U}((\cdot,\cdot);x))$. Of course, $\int \tilde{H}(y,x)d\lambda_x(y)$ is just $L^2(\lambda_x, K(x))$, which we identify also with $L^2(\lambda_x)\otimes K(x)$. Now we let x vary and choose for a fundamental family of vector fields for $\tilde{H}$, the family of fields ξ given by the formula $\xi(y,x) = ((y,x), f(y,x)\eta_n(x))$, where f runs over a countable dense subset of $C_c(\Gamma)$, and η_n comes from a fundamental sequence defining the direct integral representation of ρ. Then the collection of all such sections defines a Borel structure on $\tilde{H}$. The family $\{\mathrm{Ind}(\rho_x)\}_{x\in X}$ is now easily seen to be Borel and its direct integral, $\mathrm{Ind}(\rho)$, acting on

$$\begin{aligned}\int_{(y,x)\in\Gamma}^{\oplus} \tilde{H}(y,x)d\nu^{-1} &= \int_{x\in X}^{\oplus}\int_{s^{-1}(x)}^{\oplus} \tilde{H}(y,x)d\lambda_x d\mu(x)\\ &= \int_{x\in X}^{\oplus} L^2(\lambda_x, K(x))d\mu(x),\end{aligned}$$

where $\nu^{-1} = \int \lambda_x d\mu(x)$, is given by the formula

$$(\mathrm{Ind}(\rho)(f)\xi)(y,x) = \int \rho_x \circ \sigma_{(x,z)}(f(y,z))\xi(z,x)d\lambda^x(x,z). \tag{7}$$

When the measure μ is **quasi-invariant**, then there is a formula for $\mathrm{Ind}(\rho)$ that we will also use and which we now describe. It is the **left regular representation** determined by ρ and μ. To define it, we form the Hilbert bundle H_1 on X whose fibers are $H_1(x) = L^2(\lambda^x) \otimes K(x)$ and we identify this space with $L^2(\lambda^x, K(x))$. A family of sections defining the Hilbert bundle structure on H_1 is obtained by taking a dense sequence of functions $\{f_n\}$ in $C_c(\Gamma)$, a fundamental sequence of sections $\{\eta_n\}$ of the bundle defining $\int_X^\oplus K(x)d\mu(x)$, and letting $\xi_{n,m}$ be the section of H_1 whose value at $x \in X$ is the restriction of the $K(x)$-valued function on Γ, $f_n(\cdot)\eta_m(x)$, to $r^{-1}(x)$. To make explicit the fact that at each $x \in X$, a section ξ of H_1 evaluated at x is a function on $r^{-1}(x)$, we shall write $\xi(x;\cdot)$ or $\xi(x;(x,y))$, when we want to emphasize the variable y, for $\xi(x)$. We define $\pi_{1x}\colon A(x) \to B(H_1(x))$ to be $I \otimes \rho_x$. Thus, at the level of sections, π_{1x} is given by the formula $(\pi_{1x}(a)\xi)(x;(x,y)) = \rho_x(a)\xi(x;(x,y))$, and from this it is clear that $\{\pi_{1x}\}$ is a Borel family. The representation U_1 is simply left translation: for a section ξ of H_1, $(U_1(x,y)\xi(y))(x;(x,z)) = \xi(y;(x,y)^{-1}(x,z)) = \xi(y;(y,z))$. A calculation reveals that $(\{\pi_{1x}\}, U_1)$ is a covariant pair and that its integrated form, λ, acting on $\int_X^\oplus H_1(x)d\mu(x) = \int_X^\oplus L^2(\lambda^x, K(x))d\mu(x)$, is given by the formula

$$\begin{aligned}
&(\lambda(f)\xi)(x;(x,y)) \\
&= \left(\int U_1(x,z)\pi_{1z}(f(x,z))\Delta^{-1/2}(x,z)\xi(z)d\lambda^x(x,z)\right)(x,y) \\
&= \int \pi_{1x} \circ \sigma_{(x,z)}(f(x,z))\Delta^{-1/2}(x,z)(U_1(x,z)\xi(z))(x;(x,y))d\lambda^x(x,z) \\
&= \int \rho_x \circ \sigma_{(x,z)}(f(x,z))\Delta^{-1/2}(x,z)\xi(z;(z,y))d\lambda^x(x,z).
\end{aligned} \tag{8}$$

It now is a straightforward calculation to show that the map $W\colon \int \tilde{H}(y,x)d\nu^{-1} \to \int H_1(x)d\mu(x)$ given by the formula $(W\xi)(x;(x,y)) = \xi(y,x)\Delta^{-1/2}(x,y)$ is a Hilbert space isomorphism such that $W\,\mathrm{Ind}(\rho)(f)W^{-1} = \lambda(f)$ for all $f \in C_c(A \times_\sigma \Gamma)$.

The **reduced** C^*-norm on $C_c(A \times_\sigma \Gamma)$, $\|\cdot\|_{\mathrm{red}}$, is defined by the formula $\|f\|_{\mathrm{red}} = \sup\{\|\mathrm{Ind}(\rho)(f)\| \mid \rho$ — a representation of $C_0(A)\}$. Evidently, $\|f\|_{\mathrm{red}} \leq \|f\|$, for all $f \in C_c(A \times_\sigma \Gamma)$. The closure of $C_c(A \times_\sigma \Gamma)$ in this norm is denoted $C^*_{\mathrm{red}}(A \times_\sigma \Gamma)$; it is a quotient of $C^*(A \times_\sigma \Gamma)$. However, when the groupoid Γ is amenable, as we are assuming in our main results, the two norms coincide and so, therefore, do the C^*-algebras. The proof of this is essentially the same as for groupoids by themselves (see [14, Proposition II.3.2], and see also Theorem 3.6 in [16]).

The reduced norm is really the norm that we give $C^*(A\times_\sigma\Gamma)$, when we think of $C^*(A\times_\sigma\Gamma)$ in terms of Fell bundles. To see this, note first that if ρ_0 is any faithful representation of $C_0(A)$, then $\mathrm{Ind}(\rho_0)$ is faithful on $C^*_{\mathrm{red}}(A\times_\sigma\Gamma)$ by Theorem 4.3 in [16]. (In our setting, this can also be deduced directly, along lines of argument found in Sections 3 and 4 of Chapter II in [14] without having to invoke the technology developed in [16].) Now recall how the norm is defined on $C^*(A\times_\sigma\Gamma)$ when this algebra is viewed in terms of Fell bundles. We build the Hilbert C^*-module bundle V over $\Gamma^{(0)}$. When the bundle is $A\times_\sigma\Gamma$, the fiber over $u\in\Gamma^{(0)}$ is none other than $L^2(\lambda_u)\otimes A(u)$, viewed as the direct sum of copies of $A(u)$ with the direct sum inner product. Then, as we noted earlier, in Lemma 2, $C_0(V)$ is a Hilbert C^*-module over $C_0(A)$. The algebra $C_c(A\times_\sigma\Gamma)$, then, is represented by bounded adjointable operators in $\mathcal{L}(C_0(V))$ and the norm on $C_c(A\times_\sigma\Gamma)$ is that which is inherited from $\mathcal{L}(C_0(V))$. Given the form of V and the form of the action of $C_c(A\times_\sigma\Gamma)$ on $C_0(V)$, it is an easy calculation to discover that $\mathrm{Ind}(\rho_0)$ is none other than the representation of $C_c(A\times_\sigma\Gamma)$ that is induced, in the sense of Rieffel [17], from the representation ρ_0 of $C_0(A)$ via the Hilbert C^*-module $C_0(V)$. It follows immediately from the equation $\|f\|_{\mathrm{red}}=\|\mathrm{Ind}(\rho_0)(f)\|$, $f\in C_c(A\times_\sigma\Gamma)$, that $\|\cdot\|_{\mathrm{red}}$ is the norm on $C_c(A\times_\sigma\Gamma)$ defined in the context of Fell bundles. We summarize our discussion of these points in the following lemma.

LEMMA 10: *Under our assumption that Γ is amenable, the C^*-algebra $C^*(A\times_\sigma\Gamma)$ coincides with the reduced C^*-algebra, $C^*_{\mathrm{red}}(A\times_\sigma\Gamma)$ and the norm coincides with the norm it receives when represented on $C_0(V)$.*

PROPOSITION 11: *For each $f\in C_c(A\times_\sigma\Gamma)$, the infinity norm of f on Γ, $\|f\|_\infty$ is dominated by the C^*-norm $\|f\|$. Thus, elements in $C^*(A\times_\sigma\Gamma)$ may be viewed as C_0-sections of the bundle $A\times_\sigma\Gamma$.*

Proof: Choose a point $(x_0,y_0)\in\Gamma$ where $\|f\|_\infty=\|f(x_0,y_0)\|$ and choose a representation π of the C^*-algebra $A(y_0)$ on a Hilbert space H_π such that $\|f(x_0,y_0)\|=\|\pi(f(x_0,y_0))\|$. View π as a representation of $C_0(A)$. Then using equation (6), it is easy to see that $\|\mathrm{Ind}(\pi)(f)\|=\|\pi(f(x_0,y_0))\|=\|f(x_0,y_0)\|$. Since $\|\mathrm{Ind}(\pi)(f)\|$ is dominated by the C^*-norm of f, the proof is complete. ∎

Note that we are not assuming that $X = \Gamma^{(0)}$ is compact nor that the fiber algebras $A(x)$ are unital. Therefore, we are unable to conclude that $C^*(\Gamma)$ is contained in $C^*(A \times_\sigma \Gamma)$. However, the argument given in Lemma 4.6 of [15] shows that $C^*(\Gamma)$ is contained in the multiplier algebra of $C^*(A \times_\sigma \Gamma)$. Moreover, for $f \in C^*(\Gamma)$, and $g \in C^*(A \times_\sigma \Gamma)$,

$$f * g(x,y) = \sum_z f(x,z)g(z,y).$$

Thus we may (and will) form such products freely without special announcement.

We are now ready to attack the inclusion (2). From Proposition 11 every element of $C^*(A \times_\sigma \Gamma)$ is a C_0-section of $A \times_\sigma \Gamma$. The sections that are in B determine the bundle $F(B)$. It is evident that $B \subseteq C_0(F(B)) \cap C^*(A \times_\sigma \Gamma)$, and Proposition 11 implies that $C_0(F(B)) \cap C^*(A \times_\sigma \Gamma)$ is closed. What is not clear is that B contains any **compactly supported** sections at all. To establish this, recall that an (open) Γ-set is an open subset U of Γ such that r and s are one-to-one on U. Thus a Γ-set U is just the graph of a partially defined transformation τ whose domain is $r(U)$ and whose range is $s(U)$; and every such τ whose graph is contained in Γ is a Γ-set. We shall denote Γ-sets by τ and we write $\Omega(\Gamma)$ for the collection of all open Γ-sets τ with the property that the closure of τ is compact (and still a Γ-set that need not be open). It is shown in [8] that $\Omega(\Gamma)$ covers Γ. Let N_c denote the set of (complex valued) functions f on Γ such that the (closed) support of f is compact and contained in some $\tau \in \Omega(\Gamma)$. The notation is to indicate **compact normalizer**. The reason for this is that if $f \in N_c$, and if $d \in C_0(A)$, then $f * d * f^*$ also lies in $C_0(A)$, i.e., f normalizes $C_0(A)$ (in a weak sense.) As in [11] (see the proof of Proposition 4.4), given $g \in N_c$, we define Ψ_g: $C^*(A \times_\sigma \Gamma) \to C^*(A \times_\sigma \Gamma)$ by the formula

$$\Psi_g(f)(x,y) = \mathbf{P}(f * g^*) * g(x,y) = f(x,y)|g(x,y)|^2, \tag{9}$$

where $\mathbf{P}$ denotes the conditional expectation from $C^*(A \times_\sigma \Gamma)$ onto $C_0(A)$ given by the formula

$$\mathbf{P}(f)(\gamma) = \begin{cases} f(\gamma), \gamma \in \Gamma^{(0)}, \\ 0, \text{otherwise.} \end{cases}$$

(The fact that $\mathbf{P}$ is a bonefide conditional expectation is easy to see using Proposition 11. Of course, the $C_0(A)$-valued inner product on $C_0(V)$ is really given by the formula $\langle \xi_f, \xi_g \rangle(u) = \mathbf{P}(f^* * g)(u)$ where ξ_f and ξ_g are the sections of V determined by functions f and g in $C_c(A \times_\sigma \Gamma)$.) If g is supported on a $\tau \in \Omega(\Gamma)$, then so is $\Psi_g(f)$, for all $f \in C^*(A \times_\sigma \Gamma)$. The argument given on pages 324–5 in [11] shows that given $g \in N_c$ and $f \in C^*(A \times_\sigma \Gamma)$, one can find functions

$d_1, d_2, \ldots, d_n \in C_0(X)$, depending on f and g, so that $\sum d_i^* d_i \le 1$ and so that $\Psi_g(f) = \sum d_i^* * f * (g^* * d_i * g)$. Writing Ψ_{dg} for the map defined by the sum, it is shown that Ψ_{dg} is bounded by $\|g\|^2$. The point of introducing Ψ_{dg} is that it maps B into B because B is a bimodule over $C_0(A)$ and therefore a bimodule over $C_0(X)$, since $C_0(X)$ is contained in the multiplier algebra of $C_0(A)$. Thus since for any prescribed f we can find an n-tuple d such that $\Psi_g(f) = \Psi_{dg}(f)$, we see that there are plenty of compactly supported sections of $F(B)$ in B. We must show that every compactly supported section of $F(B)$ is contained in B. To this end note that the argument on page 325 of [11] shows that every Ψ_g, $g \in N_c$, maps B into B. Also note that every compactly supported section f of $F(B)$ is some finite $f = \sum g_i$, with $g_i \in C_c(F(B))$ and supported on a set in $\Omega(\Gamma)$. Indeed, since $\Omega(\Gamma)$ covers Γ, and the support of f is compact, we can find a finite cover $V_1, V_2, \ldots, V_n$ of $\mathrm{supp}(f)$ coming from $\Omega(\Gamma)$. Choose functions $h_1, h_2, \ldots, h_n \in C_c(\Gamma)$ that form a partition of unity subordinate to $V_1, V_2, \ldots, V_n$ and set $g_i = f \cdot h_i$ (pointwise product). Then the g_i have the desired properties. So, we need only show that each compactly supported section of $F(B)$ with closed support contained in some $\tau \in \Omega(\Gamma)$ necessarily lies in B. Fix τ and an $f_0 \in C_c(F(B))$ with $\overline{\mathrm{supp}(f_0)} \subseteq \tau$. Then since B constitutes a total family of sections of $F(B)$ in the sense of [3, 10.2.1] and since the closure of τ is a compact Γ-set, we can find a sequence $\{f_n\} \subseteq B$ that converges uniformly to f_0 on τ by [3, 10.2.5] or [6, II.14.1]. Once more, choose an open cover $V_1, V_2, \ldots, V_n$ of $\mathrm{supp}(f_0)$, with $V_i \subseteq \tau$, choose functions $h_1, h_2, \ldots, h_n \in C_c(\Gamma)$ that form a partition of unity subordinate to $V_1, V_2, \ldots, V_n$, and for $f \in C^*(A \times_\sigma \Gamma)$, set $\Psi(f) = \sum \Psi_{h_i}(f)$. Then since the h_i are all supported in τ, equation (9) shows that each $\Psi(f_n)$ is supported on τ and that this sequence converges uniformly on τ to $\Psi(f_0) = f_0$. Since, as we have shown, Ψ maps B into B, we have produced a sequence of sections in B, all supported on τ, that converges uniformly to f_0. However, such a sequence clearly converges to f_0 in the I-norm (5) and therefore in the C^*-norm. Thus, $f_0 \in B$.

To complete the proof of Theorem 9, we need only show that $C_c(F(B))$ is dense in $C_0(F(B)) \cap C^*(A \times_\sigma \Gamma)$. Suppose not. Then by Lemma 3.9 in [13], we can find a representation π of $C^*(A \times_\sigma \Gamma)$ on a Hilbert space H and vectors ξ and η in H such that the vector functional ϕ: $f \to \langle \pi(f)\xi, \eta \rangle$ annihilates $C_c(F(B))$ but does not annihilate $C_0(F(B)) \cap C^*(A \times_\sigma \Gamma)$. Writing π in its disintegrated form, (4), we may express ϕ as follows:

$$\phi(f) = \int \langle U(x,y)\pi_y(f(x,y))\xi(y), \eta(x) \rangle d\nu_0(x,y),$$

where $\nu_0 = \Delta^{-1/2} \cdot d\nu$. Let $\{f_i\}$ be a sequence in $C_c(\Gamma)$ defining an approximate invariant mean on $L^\infty(\nu)$, i.e., satisfying the conditions of Definition 3.1 in [14], and set

$$(10) \quad \phi_i(f) = \int_\Gamma \left(\sum_z f_i(x,z)\overline{f_i(y,z)} \right) \langle U(x,y)\pi_y(f(x,y))\xi(y), \eta(x)\rangle d\nu_0(x,y).$$

Then the functionals ϕ_i converge pointwise to ϕ. Since ϕ annihilates $C_c(F(B))$, $\int \langle U(x,y)\pi_y(f(x,y))\xi(y), \eta(x)\rangle d\nu_0(x,y) = 0$ for every $f \in C_c(F(B))$. Since $C_c(F(B))$ is a bimodule over $C_c(\Gamma)$ under **pointwise** multiplication, we conclude that for every $g \in C_c(\Gamma)$,

$$\int g(x,y)\langle U(x,y)\pi_y(f(x,y))\xi(y), \eta(x)\rangle d\nu_0(x,y) =$$
$$\int \langle U(x,y)\pi_y(g(x,y)f(x,y))\xi(y), \eta(x)\rangle d\nu_0(x,y) = 0.$$

This formula persists for all $g \in L^\infty(\nu_0)$, since the function

$$(x,y) \to \langle U(x,y)\pi_y(f(x,y))\xi(y), \eta(x)\rangle$$

is in $L^1(\nu_0)$, and every function in $L^\infty(\nu_0)$ is the bounded, pointwise limit of a sequence from $C_c(\Gamma)$. Since the function $(x,y) \to \left(\sum_z f_i(x,z)\overline{f_i(y,z)}\right)$ is in $L^\infty(\nu_0)$, it follows that each functional ϕ_i annihilates $C_c(F(B))$. On the other hand, since ϕ does not annihilate $C_0(F(B)) \cap C^*(A \times_\sigma \Gamma)$, there is a section f_0 in this space and an i_0 such that $\phi_{i_0}(f_0) \neq 0$.

Let ρ denote $\pi|C_0(A)$ where, recall, $C_0(A)$ is viewed as the diagonal subalgebra of $C^*(A \times_\sigma \Gamma)$, i.e., $C_0(A)$ is the subalgebra of $C^*(A \times_\sigma \Gamma)$ consisting of all sections supported on $\Gamma^{(0)}$. Form $\mathrm{Ind}(\rho)$ using the formulation given in equation (8), i.e., identify $\mathrm{Ind}(\rho)$ with the representation λ in that equation. Then, a straightforward calculation shows that if we set

$$\xi'(x;(x,y)) = \Delta(x,y)^{1/2}\overline{f_{i_0}(x,y)}U(y,x)\xi(x)$$

and

$$\eta'(x;(x,y)) = \Delta(x,y)^{1/2}\overline{f_{i_0}(x,y)}U(y,x)\eta(x),$$

then ϕ_{i_0} is given by the formula

$$\phi_{i_0}(f) = \mathrm{Ind}(\rho)(f),$$

$f \in C^*(A \times_\sigma \Gamma)$. Thus, we conclude that the σ-weak closures of $\mathrm{Ind}(\rho)(C_c(F(B)))$ and $\mathrm{Ind}(\rho)(C_0(B) \cap C^*(A \times_\sigma \Gamma))$ are distinct. However, this runs contrary to Theorem 15.18 of [7], as we now show.

Let $M(x)$ be the σ-weak closure of $\rho_x(A(x)) = \pi_x(A(x))$ acting on the Hilbert space $H(x)$. The collection $\{M(x)\}_{x\in X}$ is a measurable field of von Neumann algebras whose direct integral $M = \int_X^\oplus M(x)d\mu(x)$ acts on H and is the σ-weak closure of $\rho(A)$. Further, since $(\{\pi_x\}, U)$ is a covariant pair, each automorphism $\sigma_{(x,y)}$ extends to be a σ-weakly continuous isomorphism from $M(y)$ to $M(x)$ and the family $\{\sigma_{(x,y)}\}$ satisfies the measurability hypotheses in [7, p. 4]. From the form of $\mathrm{Ind}(\rho)$ spelled out in equation (8), it is clear that the σ-weak closure of $\mathrm{Ind}(\rho)(C^*(A \times_\sigma \Gamma))$ is the von Neumann algebra crossed product $M \bowtie \Gamma$ of [7, Definition 2.1], also denoted by $\tilde{M}$. (The cocycle c used there is taken to be identically 1 here and will be omitted from the discussion.) Since Γ is measurewise amenable with respect to every quasi-invariant measure on X, Γ is hyperfinite when viewed as a measured equivalence relation [11, Theorem 3.3]. Thus, we are in a position to use Theorem 15.18 of [7].

Let $\mathfrak{B}_1$ be the σ-weak closure of $\mathrm{Ind}(\rho)(C_c(F(B)))$ and let $\mathfrak{B}_2$ be the σ-weak closure of $\mathrm{Ind}(\rho)(C_0(B) \cap C^*(A \times_\sigma \Gamma))$. Then from what we showed above, $\mathfrak{B}_1 \subsetneq \mathfrak{B}_2$. Each of $\mathfrak{B}_1$ and $\mathfrak{B}_2$ is a bimodule over M viewed as a subalgebra of $M \bowtie \Gamma$. (In [7], when M is viewed this way, it is written $I(M)$.) By Theorem 15.18 of [7], there is, for $i = 1, 2$, a projection valued function P_i on Γ such that $P_i(x, y)$ lies in the center of $M(y)$, for ν-almost all (x, y), and such that $\mathfrak{B}_i = P_i \circ (M \bowtie \Gamma)$, where, for $T \in M \bowtie \Gamma$, $P_i \circ T$ denotes the Schur product of P_i and T, i.e., pointwise product. Since $\mathfrak{B}_1 \subsetneq \mathfrak{B}_2$, $P_1(x, y) \leq P_2(x, y)$, a.e. ν, and because the inclusion is strict, $P_1(x, y) \lneq P_2(x, y)$ for (x, y) in a set of positive ν-measure. However, for ν-almost all (x, y), both $P_1(x, y) \cdot M(y)$ and $P_2(x, y) \cdot M(y)$ coincide with the σ-weak closure of $F(x, y)$ in $M(y)$ because for each $(x, y) \in \Gamma$, $\{f(x, y) | f \in C_c(F(B))\} = F(B)(x, y) = \{f(x, y) | f \in C_0(F(B)) \cap C^*(A \times_\sigma \Gamma)\}$ (no closures are necessary). Indeed, by the theorem of Douady and Dal Soglio-Hérault [5, Appendix], given any $\xi \in F(B)(x, y)$, there is a continuous section f of $F(B)$ such that $f(x, y) = \xi$. By Proposition 10.1.9 of [3] we may assume that f has compact support, i.e., that $f \in C_c(F(B))$. Thus $P_1(x, y) \cdot M(y) = P_2(x, y) \cdot M(y)$ a.e. ν. This contradiction completes the proof of Theorem 9, and with it, the first assertion of Theorem 1 is proved.

The proof of the second assertion of Theorem 1, however, is easy: If $F \in \mathcal{A}$, then it is clear from the definition of the product in $C^*(E)$ that $B(F)$ is a subalgebra of $C^*(E)$. For the converse implication, suppose F is such that $B(F)$ is a subalgebra and choose a point $(\xi, \eta) \in F^{(2)} := (F \times F) \cap E^{(2)}$. As we noted in the preceding paragraph, we may find sections $f, g \in C_c(F)(\subseteq B(F))$ such that $f(p(\xi)) = \xi$ and $g(p(\eta)) = \eta$. Furthermore, from the analysis that went into proving the inclusion (2), we may assume that f and g are supported on Γ-sets.

It follows that $\xi\eta = fg(p(\xi)p(\eta))$. Since $fg \in B(F)$ by hypothesis, we conclude that $\xi\eta \in F$. Thus $F \in \mathcal{A}$. This completes the proof of Theorem 1.

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